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$SU_q(1, 1)$ and the relativistic oscillator

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Abstract. It is shown that the generalization of the quantum harmonic oscillator to the case of the relativistic configurational space is a q-oscillator. The corresponding group of dynamical symmetry is the quantum group $SU_q(1, 1)$. The deformation parameter being $q = e^{-\omega/\omega_0}$ where $\hbar\omega_0 = 4mc^2$ and ω is a frequency of the oscillator. The deformed creation and annihilation operators are finite difference ones. The corresponding deformation of the Heisenberg–Weyl group and new coherent states are also considered.

1. Introduction

The purpose of this paper is to give an example of an exactly solvable problem possessing at the same time quantum symmetry. In the last few years quantum groups have attracted great interest from both physicists and mathematicians. Among the important problems is the search for real physical systems with q-symmetry. One tries to find an exactly solvable problem in which the corresponding symmetry manifests itself as the quantum Lie algebra and the wavefunctions realize its representations. A series of papers was devoted to the q-oscillator (Biedenharn 1989, Macfarlane 1989, Kagramanov *et al* 1990).

The quantum deformation of Lie algebras emerged initially as the basic algebraic structure connected with the Yang–Baxter equation and the quantum inverse method (Faddeev 1984, Faddeev *et al* 1989, Manin 1987, Kulish and Sklyanin 1982, Kulish and Reshetikhin 1983, Chaichian and Kulish 1990, Chaichian *et al* 1990). The author of the present paper agrees with the point of view expressed by Carow-Watamura *et al* (1990; see also Bayen *et al* 1978) that this new type of deformation based on the Hopf algebra structure of the functions over groups must be considered in the general framework of deformations of the physical theories and models. Between other well known deformations are special relativity as deformed Galilean relativity with the velocity of light c as the deformation parameter or quantum mechanics as deformed classical mechanics with the Planck constant $\hbar$ as the deformation parameter.

Further, we shall show here that in our concrete relativistic deformation of quantum mechanics the parameter c of the relativistic deformation is at the same time the q-deformation parameter of the group of dynamical symmetry. In other words, in this approach the q-deformation is the relativistic effect which disappears in the non-relativistic limit.

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We consider the relativistic deformation of the Schrödinger equation based on the Gelfand–Graev–Shapiro transformation, i.e. the Fourier expansion in relativistic 'deformed plane waves' (Kadyshevsky *et al* 1968, Mir-Kasimov 1966)

$$\langle \boldsymbol{r}|\boldsymbol{p}\rangle = \left(\frac{p_0 - \boldsymbol{p}\boldsymbol{n}}{mc}\right)^{-1-ir(mc/\hbar)} \qquad (1)$$

$$\boldsymbol{r} = r\boldsymbol{n} \qquad \boldsymbol{n}^2 = 1 \qquad 0 < r < \infty.$$

The space of vectors $\boldsymbol{r}$ will be called the relativistic configurational space or r-space. We shall see that the gometry of this space is the deformed Euclidean geometry of the usual three-dimensional configurational space. The variable r is the relativistic invariant and can be expressed in terms of the eigenvalues of the Casimir operator of the Lorentz group $\hat{C} = \boldsymbol{N}^2 - \boldsymbol{L}^2$, where $\boldsymbol{N}$ and $\boldsymbol{L}$ are boost and rotation generators:

$$\hat{C}\langle \boldsymbol{r}|\boldsymbol{p}\rangle = \left[\left(\frac{\hbar}{mc}\right)^2 + r^2\right]\langle \boldsymbol{r}|\boldsymbol{p}\rangle. \qquad (2)$$

The four-momentum vector p_μ $(\mu = 0, 1, 2, 3)$ of the free relativistic particle with mass m belongs to the mass shell, i.e. the upper sheet of the hyperboloid

$$p_0^2 - \boldsymbol{p}^2 = m^2c^2 \qquad (3)$$

or p-space of Lobachevsky. The group of motions of this space is the Lorentz group.

In the non-relativistic limit

$$|\boldsymbol{p}| \ll mc \qquad r \gg \frac{\hbar}{mc} \qquad p_0 \approx mc + \frac{\boldsymbol{p}^2}{2mc} \qquad (4)$$

the deformed plane wave (1) goes over into the usual plane wave:

$$\begin{aligned}\langle \boldsymbol{r}|\boldsymbol{p}\rangle &= \exp\left[-\left(1 + ir\frac{mc}{h}\right)\ln\left(\frac{p_0 - \boldsymbol{p}\boldsymbol{n}}{mc}\right)\right] \\ &\approx \exp\left[-\left(1 + ir\frac{mc}{h}\right)\ln\left(1 - \frac{\boldsymbol{p}\boldsymbol{n}}{mc} + \frac{\boldsymbol{p}^2}{2m^2c^2} + \dots\right)\right] \\ &\approx \exp\left(ir\frac{\boldsymbol{p}\boldsymbol{n}}{h}\right) = \exp\left(i\frac{\boldsymbol{p}\boldsymbol{x}}{h}\right).\end{aligned} \qquad (5)$$

2. Relativistic quantum mechanics

Relativistic quantum mechanics in r-space was developed in several papers (Atakishiyev *et al* 1980, 1985, 1986, Kadyshevsky *et al* 1968, 1969, Freeman *et al* 1969, Kagramanov *et al* 1990). As in the non-relativistic limit (4) this theory reduces to usual quantum mechanics. For the sake of clarity of presentation we write down the necessary non-relativistic formulae. The Schrödinger equation is the second-order differential equation

$$\begin{aligned}&(\hat{H}_0^{nr} + V(r) - E_q)\psi(\boldsymbol{r}) = 0 \\ &\hat{H}_0^{nr} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\Delta_{\vartheta,\varphi}}{r^2}\right) E_q = \frac{\boldsymbol{q}^2}{2m}.\end{aligned} \qquad (6)$$

The wavefunction in momentum space $\psi(\boldsymbol{p})$ is connected to $\psi(\boldsymbol{r})$ by the Fourier transformation

$$\psi(\boldsymbol{r})=\frac{1}{(2\pi)^3}\int e^{i\boldsymbol{p}\boldsymbol{r}/h}\,\psi(\boldsymbol{p})\,d^3\boldsymbol{p}. \tag{7}$$

Equation (6) in momentum space takes the form

$$\psi(\boldsymbol{p})=(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q})+\frac{1}{(2\pi)^3}G_q^{nr}(p)\int V(\boldsymbol{p}-\boldsymbol{k})\psi(\boldsymbol{k})\,d\boldsymbol{k}. \tag{8}$$

Momentum space over which the integration in (7) and (8) is carried out is the three-dimensional Euclidean space. More important is that the plane wave $e^{i\boldsymbol{p}\boldsymbol{x}/h}$ is naturally connected with the geometry of momentum space. It is the matrix element of the group of translations, and at the same time it is the generating function for the matrix elements of the unitary irreducible representations of the group of motions of momentum space (i.e. the three-dimensional Euclidean group) on a spherical basis. This fact is reflected in the well known relation

$$e^{i\boldsymbol{p}\boldsymbol{x}/h}=\sum(2l+1)\,i^l j_l\left(\frac{px}{\hbar}\right)P_l\left(\frac{\boldsymbol{p}\boldsymbol{x}}{px}\right) \tag{9}$$

where the spherical Bessel functions $j_l(px/h)=\sqrt{\pi h/2px}\,J_{l+1/2}(px/h)$ are the matrix elements corresponding to the subgroup of translations. The absolute value x of the $\boldsymbol{x}$-vector is an invariant of the group of motions of p-space or eigenvalue of its Casimir operator:

$$(\boldsymbol{x}^2)e^{i\boldsymbol{p}\boldsymbol{x}/h}=-\hbar^2\Delta_p\,e^{i\boldsymbol{p}\boldsymbol{x}/h}=r^2\,e^{i\boldsymbol{p}\boldsymbol{x}/h}\qquad \boldsymbol{x}=i\hbar\frac{\partial}{\partial\boldsymbol{p}}. \tag{10}$$

If the two-body problem is considered then $\boldsymbol{p}$, $\boldsymbol{k}$ and $\boldsymbol{x}$ in (6) and (8) are the relative momentum and relative distance of the two particles correspondingly.

It is worthwhile mentioning here that the group of motions of the momentum space does not have such a transparent physical sense as the spacetime group of invariance of non-relativistic quantum mechanics which is the Galilean group. Let us also stress that r is Galilean invariant.

We can derive the relativistic Schrödinger equation passing from the Euclidean non-relativistic momentum space to the Lobachevsky space (3). So we come to the quasipotential equation (see Kadyshevsky *et al* 1968 and references therein)

$$\psi(\boldsymbol{p})=(2\pi)^3\sqrt{1+\boldsymbol{p}^2}\,\delta(\boldsymbol{p}-\boldsymbol{q})+\frac{1}{(2\pi)^3}G_q(p)\int V(\boldsymbol{p},\boldsymbol{k};E_q)\psi(\boldsymbol{k})\,d\Omega_k \tag{11}$$

where

$$d\Omega_k=\frac{d\boldsymbol{k}}{\sqrt{1+\boldsymbol{k}^2}} \tag{12}$$

is the volume element of Lobachevsky space. In (10), (11) and in what follows we use the unit system in which $\hbar=c=m=1$

$$G_q(p)=(2q_0-2p_0+i\varepsilon)^{-1}. \tag{13}$$

To obtain the relativistic Schrödinger equation in $\boldsymbol{r}$-space we must expand $\psi(\boldsymbol{p})$ in relativistic plane waves (1):

$$\psi(\boldsymbol{r})=\frac{1}{(2\pi)^3}\int\langle\boldsymbol{r}|\boldsymbol{p}\rangle\psi(\boldsymbol{p})\,\mathrm{d}\Omega_p. \tag{14}$$

Let us consider this in more detail. The relativistic plane waves are the eigenfunctions of the Casimir operator (2) and therefore are generating functions for the matrix elements of the principal series of the unitary irreducible representations of the Lorentz group:

$$\langle\boldsymbol{r}|\boldsymbol{p}\rangle=\sum_{l=0}^{\infty}(2l+1)\mathrm{i}^l p_l(\cosh\chi,r)P_l\left(\frac{\boldsymbol{pn}}{p}\right) \tag{15}$$

where

$$p_l(\cosh\chi,r)=(-1)^l\sqrt{\frac{\pi}{2\sinh\chi}}\,\frac{\Gamma(\mathrm{i}r+l+1)}{\Gamma(\mathrm{i}r+1)}\,P^{-1/2-l}_{-1/2+\mathrm{i}r}(\cosh\chi) \tag{16}$$

and the polyspherical coordinates

$$p_0=\cosh\chi\qquad \boldsymbol{p}=\boldsymbol{n}_p\sinh\chi$$

where introduced.

The difference between the right-hand sides of (15) and (9) is in their radial parts. In (15) we have Legendre functions which correspond to boosts, instead of the spherical Bessel functions in (9) corresponding to shifts. It is instructive to compare the radial parts of (9) and (15):

$$j_l(pr)=\sqrt{\frac{\pi}{2pr}}\,\frac{\mathrm{e}^{-\mathrm{i}pr}\,(\frac{1}{2}pr)^{l+1/2}}{\Gamma(l+\frac{3}{2})}\,{}_1F_1(l+1;2l+2;-2\,\mathrm{i}pr) \tag{17}$$

$$p_l(\cosh\chi,r)=\sqrt{\frac{\pi\,\mathrm{e}^{-x}}{22\sinh\chi r}}\,\frac{(-1)^l\Gamma(\mathrm{i}r+l+1)}{\Gamma(l+\frac{3}{2})\Gamma\,\mathrm{i}r+1)}\left(\frac{\sinh\chi\,\mathrm{e}^{-\chi}}{2}\right)^{l+1/2}$$
$$\times{}_2F_1(-\mathrm{i}r+l+1,l+1;2l+2;2\sinh\chi\,\mathrm{e}^{-x}). \tag{18}$$

We see that as a consequence of the relativistic deformation the r-dependence was displaced from the argument of the hypergeometric function in (17) to its parameter in (18). As the consequence the number of the left subscripts of ${}_pF_q$ increased from 1 to 2. This feature carries a general character. For example, the solution of the relativistic Coulomb problem in this approach was obtained through similar deformation of the non-relativistic Coulomb wavefunction (Freeman *et al* 1969). So the free radial wavefunction becomes the function of the 'discrete' variable (Kadyshevsky *et al* 1969). Functions of this type do not satisfy any differential equation of finite order, but only the finite-difference or recurrence equation.

It is easy to verify that the relativistic plane wave obeys the differential-difference Schrödinger equation

$$(H_0-p_0)\langle\boldsymbol{r}|\boldsymbol{p}\rangle=0 \tag{19}$$

where the Hamiltonian operator has the form

$$H_0=\cosh\mathrm{i}\frac{\partial}{\partial r}+\frac{\mathrm{i}}{r}\sinh\mathrm{i}\frac{\partial}{\partial r}-\frac{\Delta_{\upsilon,\varphi}}{r^2}\,\mathrm{e}^{\mathrm{i}\partial/\partial r}. \tag{20}$$

The relations (19) and (20) can be considered as the solution of the old problem of extracting the square root in the expression for the relativistic energy:

$$p_0=\sqrt{\boldsymbol{p}^2+m^2}. \tag{21}$$

Of course this statement makes sense if we prove that the scheme based on (20) does really work, which is the case. The formalism of finite-difference quantum mechanics was developed by Kadyshevsky *et al* (1968) and Freeman *et al* (1969) and carries many features of differential formalism of non-relativistic quantum mechanics. The interacting particles are described by the equation

$$(H_0+V(\boldsymbol{r})-p_0)\psi(\boldsymbol{r})=0. \tag{22}$$

The potential $V(\boldsymbol{r})$ can be obtained as the quasipotential in the field-theoretic formalism of the quasipotential (11), or introduced phenomenologically. The scattering theory based on partial shifts was built up. The theory of functions of discrete variables, generalizing the elementary and the most important special functions for the case of finite-difference calculus, was developed. These functions are deformations of their non-relativistic analogues in the sense described previously (cf (17) and (18)). All the approximations usually used in quantum mechanics, including the variable phase approach, were also developed. All important exactly solvable cases of quantum mechanics (potential well, Coulomb potential, harmonic oscillator) are exactly solvable as well as for the case of (22). The manifold of solutions of the finite-difference equations is richer than for differential equations. For example, among the unusual solutions there exists one which can be considered as the wavefunction of the confined quark-antiquark system (Kagramanov *et al* 1989).

It is important to stress the Euclidean or deformed Euclidean character of the $\boldsymbol{r}$-space. There exist three commuting operators $\hat{p}_i$ of three momenta:

$$[\hat{p}_i,\hat{p}_j]=0 \qquad i,j=1,2,3 \tag{23}$$

$$\begin{aligned}
\hat{p}_1&=\sin\vartheta\cos\varphi(\mathrm{e}^{\mathrm{i}\partial/\partial r}-\hat{H}_0)-\mathrm{i}\left(\frac{\cos\varphi\cos\vartheta}{r}\frac{\partial}{\partial\vartheta}-\frac{\sin\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\mathrm{e}^{\mathrm{i}\partial/\partial r}\\
\hat{p}_2&=\sin\vartheta\sin\varphi(\mathrm{e}^{\mathrm{i}\partial/\partial r}-\hat{H}_0)-\mathrm{i}\left(\frac{\sin\varphi\cdot\cos\vartheta}{r}\frac{\partial}{\partial\vartheta}-\frac{\cos\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\mathrm{e}^{\mathrm{i}\partial/\partial r}\\
\hat{p}_3&=-\cos\vartheta(\mathrm{e}^{\mathrm{i}\partial/\partial r}-\hat{H}_0)+\frac{\mathrm{i}\sin\vartheta}{r}\frac{\partial}{\partial\vartheta}\mathrm{e}^{\mathrm{i}\partial/\partial r}
\end{aligned} \tag{24}$$

for which the plane waves (1) are the common eigenfunctions

$$\hat{p}_i\langle\boldsymbol{r}|\boldsymbol{p}\rangle=p_i\langle\boldsymbol{r}|\boldsymbol{p}\rangle. \tag{25}$$

This means that the deformed plane waves (1) actually describe the free relativistic motion with definite energy and momentum. But from the group-theoretic point of view we have here the deformed representation of the inhomogeneous Euclidean group. For example, the operator of angular momentum

$$M_{12}=r_1\hat{p}_1-r_2\hat{p}_1=\mathrm{i}\frac{\partial}{\partial\varphi}\mathrm{e}^{\mathrm{i}\partial/\partial r} \tag{26}$$

is a generator of rotation

$$L_{12}=\mathrm{i}\frac{\partial}{\partial\varphi} \tag{27}$$

deformed by the factor $\mathrm{e}^{\mathrm{i}\partial/\partial r}$.

3. The one-dimensional case

In the case of only one spatial component the relativistic plane wave takes the form

$$\langle x|p\rangle = (p_0 - p)^{-ix} \tag{28}$$

or, in hyperpolar coordinates,

$$p_0 = \cosh\chi \qquad p = \sinh\chi \tag{29}$$

we have the exponential function

$$\langle x|p\rangle = e^{ix\chi}. \tag{30}$$

The difference from the usual non-relativistic plane wave is that here in the exponent instead of the momentum we have its hyperbolic argument, i.e. the rapidity

$$\chi = \ln(p_0 + p). \tag{31}$$

The one-dimensional plane waves obey the completeness and orthogonality conditions

$$\frac{1}{2\pi}\int \langle x|p\rangle \, d\Omega_p \langle p|x\rangle = \delta(x - x') \tag{32}$$

$$\frac{1}{2\pi}\int \langle p|x\rangle \, dx \langle x|p'\rangle = p_0\delta(p - p') = \delta(\chi - \chi') \tag{33}$$

where

$$d\Omega_p = \frac{d \sinh\chi}{\cosh\chi} = d\chi \qquad \langle p|x\rangle = \langle x|p\rangle^*. \tag{34}$$

The free Hamiltonian and momentum operators are again the finite-difference operators

$$\hat{H}_0 = \cosh i\frac{d}{dx} \qquad \hat{p} = -\sinh i\frac{d}{dx}. \tag{35}$$

The free equation has the form

$$(\hat{H}_0 - p_0)\langle x|p\rangle = 0. \tag{36}$$

We can rewrite this equation in such a way as to make it indistinguishable from the non-relativistic Schrödinger equation. This is achieved by passing to the 'half rapidity' variable. We use the relation

$$\cosh\chi = 1 + 2\sinh^2\frac{\chi}{2}. \tag{37}$$

Now we return for the time being to the dimensional quantities and introduce the relativistic 'kinetic energy' operator $\hat{h}_0$ by the relation

$$\hat{h}_0 = \hat{H}_0 - mc^2 = 2mc^2 \sinh^2\frac{i}{2}\frac{d}{dx}. \tag{38}$$

The operator of relativistic 'kinetic momentum'

$$\hat{k} = -2\, mc \sinh\frac{i}{2}\frac{d}{dx} \tag{39}$$

is connected to the relativistic free energy operator $\hat{h}_0$ by the non-relativistic relation

$$\hat{h}_0 = \frac{\hat{k}^2}{2m}. \tag{40}$$

Finally, we come to the one-dimensional relativistic Schrödinger equation

$$(\hat{h} - e)\psi(x) = (\hat{h}_0 + V(x) - e)\psi(x) = \left(\frac{\hat{k}^2}{2m} + V(x) - \frac{k^2}{2m}\right)\psi(x) = 0 \tag{41}$$

where

$$e = \frac{k^2}{2m} \qquad k = 2mc \sinh \frac{\chi}{2}. \tag{42}$$

It is impossible to distinguish (41) from the usual Schrödinger equation until we concretize the finite-difference operator of the free energy. This is a manifestation of the close similarity of the relativistic approach we are considering and the usual quantum mechanics.

In is also worthwhile to stress that owing to the finite-difference (recurrence) character of operators this theory is very close to the theory on the lattice. In fact, all results of this paper can easily be rewritten for a lattice.

4. The relativistic linear quantum oscillator

A direct method for the relativistic generalization of the harmonic oscillator potential or the elastic force does not exist. We define the relativistic quantum oscillator by applying a number of natural requirements on (41) (see also Kim and Noz 1978, Bohm *et al* 1985, Atakishiev *et al* 1980, 1985, 1986, Mukunda *et al* 1980). It must be the exactly sovable case, the non-relativistic limit of which is the usual linear oscillator. Symmetry must exist between the equations in momentum and configurational representations. This property plays an important role in the non-relativistic case. The coherent states minimizing the corresponding uncertainties relation must exist. And, of course, any oscillator model of interest must incorporate dynamical symmetry, which is natually some deformation of the SU(1, 1) group of dynamical symmetry of the non-relativistic oscillator to which this generalized symmetry reduces in the non-relativistic limit. We shall see that for our relativistic oscillator it is the $SU_q(1, 1)$ quantum group, where the relativistic parameter of deformation q has the form

$$q = e^{-\omega h/4mc^2} \tag{43}$$

where ω is the oscillator parameter (frequency). We see that the parameter of deformation is a pure relativistic quantity and becomes unity in the non-relativistic limit.

In the foundation of our construction lies the relativistic finite-difference generalization of the well known factorization method (Basu and Wolf 1983, Moshinsky 1969, Infeld and Hull 1951).

We recall here the necessary relations connected with the finite-difference factorization method (Kagramanov *et al* 1990). We suppose that the bound-state wavefunction has the form

$$\psi_0(x) = e^{-\rho(x)} \tag{44}$$

where ρ is a positive-definite function. Then we look for the generalizations of the creation and annihilation operators $A^{\pm}$ in the form

$$A^{\pm} = \mp \frac{2\mathrm{i}\alpha(x)}{\sqrt{2}} \mathrm{e}^{\rho(x)} \sinh \frac{\mathrm{i}}{2} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{-\rho(x)} \tag{45}$$

where $\alpha(x)$ is an arbitrary function which we have to define. It can be shown that as a consequence of the more complicated rules of the finite-difference calculus, for example

$$\sinh \frac{\mathrm{i}}{2} \frac{\mathrm{d}}{\mathrm{d}x} (\varphi(x)\psi(x)) = \sinh \frac{\mathrm{i}}{2} \frac{\mathrm{d}}{\mathrm{d}x} \varphi(x) \cosh \frac{\mathrm{i}}{2} \frac{\mathrm{d}}{\mathrm{d}x} \psi(x) + \cosh \frac{\mathrm{i}}{2} \frac{\mathrm{d}}{\mathrm{d}x} \varphi(x) \sinh \frac{\mathrm{i}}{2} \frac{\mathrm{d}}{\mathrm{d}x} \psi(x) \tag{46}$$

compared with the differential calculus, we must consider the deformed commutator instead of the usual one:

$$[A^-, A^+]_{q(x)} = A^- q(x) A^+ - A^+ q^{-1}(x) A^- \tag{47}$$

where

$$\ln q(x) = 2\left[\left(\cosh \frac{\mathrm{i}}{2} \frac{\mathrm{d}}{\mathrm{d}x} \rho(x)\right) - \rho(x)\right]. \tag{48}$$

Substituting (45) and (48) into (47) we obtain

$$[A^-, A^+]_{q(x)} = -\tfrac{1}{2}\alpha(x)[\alpha_+\beta_+ + \alpha_-\beta_-] \tag{49}$$

where

$$\alpha_{\pm}(x) = \alpha(x \pm \tfrac{1}{2}) \tag{50a}$$

$$\beta_{\pm}(x) = \sinh(\rho(x+\mathrm{i}) - \rho(x)) \pm \sinh(\rho(x-\mathrm{i}) - \rho(x)). \tag{50b}$$

For the non-relativistic oscillator, the creation and annihilation operators are

$$\alpha^{\pm} = \mp \frac{1}{\sqrt{2}} \mathrm{e}^{\pm\omega x^2/2} \frac{\partial}{\partial x} \mathrm{e}^{\mp\omega x^2/2} = \frac{1}{\sqrt{2}}\left(\mp \frac{\mathrm{d}}{\mathrm{d}x} + \omega x\right) \tag{51}$$

and the corresponding commutator is equal to a constant:

$$[a^-, a^+] = \omega. \tag{52}$$

Let us accept by definition that in the case of the relativistic oscillator a similar equality holds:

$$[A^-, A^+]_{q(x)} = \text{constant}. \tag{53}$$

Considering (53) as the equation for $\rho(x)$ we find that

$$\rho(x) = \frac{\omega x^2}{2} \qquad \psi_0(x) = \mathrm{e}^{-\omega x^2/2} \tag{54}$$

which means that the ground state wavefunction for the relativistic potential coincides with the non-relativistic one. The deformation parameter $q(x)$ becomes a constant (43) and the function $\alpha(x)$ has to be equal to

$$\alpha(x) = \cos \omega x/2 \tag{55}$$

we have

$$A^{\pm} = \pm \frac{2i}{\sqrt{2}} e^{\pm\omega/8} \left(\sinh \frac{i}{2} \frac{d}{dx} \mp i \tan \frac{\omega x}{2} \cosh \frac{i}{2} \frac{d}{dx} \right). \tag{56}$$

As $q(x) = \text{constant}$ the deformed commutator (97) becomes the combination of the usual commutator and anticommutator:

$$[A^-, A^+]_q = qA^-A^+ - q^{-1}A^+A^- = \cosh \frac{\omega}{4} [A^-, A^+] - \sinh \frac{\omega}{4} \{A^-, A^+\}. \tag{57}$$

Our relativistic finite-difference creation and annihilation operators $A^{\pm}$ are actually very close to the finite-difference operators b, b^+ employed by Macfarlane (1989) with the commutation relation

$$qbb^+ - q^{-1}b^+b = q - q^{-1}. \tag{58}$$

In Macfarlane's paper the operators were introduced by definition. In our approach we can obtain them in a framework of the finite-difference factorization method. The operators b, b^+ are restored if we use the following splitting of the free Hamiltonian (40):

$$\hat{h}_0 = -2 \sinh^2 \frac{i}{2} \frac{d}{dx} = 2(1 - e^{-id/dx})(1 - e^{id/dx}) \tag{59}$$

and take the slightly modified ground state wavefunction

$$\psi_0(x) = e^{-\omega x^2/2 - 2i\omega x} \tag{60}$$

instead of (54).

We can express $A^{\pm}$ in terms of the non-relativistic creation and annihilation operators (51) using the Baker-Campbell-Hausdorf formula:

$$A^{\pm} = -\frac{i\sqrt{2}}{\cos \frac{1}{2}\omega x} \sinh \frac{i}{2} \left(\mp \frac{d}{dx} + \omega x \right) = -\frac{i\sqrt{2}}{\cos[(a^+ + a^-)/2\sqrt{2}]} \sinh \frac{i}{\sqrt{2}} a^{\pm}. \tag{61}$$

Let us introduce the operators

$$\hat{D} = -\frac{2i}{\cos \frac{1}{2}\omega x} \sinh \frac{i}{2} \frac{d}{dx} \tag{62}$$

$$\hat{T} = \frac{1}{\cos \frac{1}{2}\omega x} \cosh \frac{i}{2} \frac{d}{dx}. \tag{63}$$

We can write

$$A^{\pm} = \mp \frac{1}{\sqrt{2}} e^{\pm\omega x^2/2} \hat{\mathscr{D}} e^{\mp\omega x^2/2}. \tag{64}$$

A simple calculation gives

$$[A^-, A^+]_q = 4 \sinh \frac{\omega}{4} = 2(q^{-1} - q). \tag{65}$$

The Hamiltonian of a relativistic oscillator is written in the factorized form

$$\begin{aligned} \hat{h} &= \tfrac{1}{2}\{A^-, A^+\}_q = \tfrac{1}{2}\{e^{-\omega/4} A^-A^+ + e^{\omega/4} A^+A^-\} \\ &= e^{\omega/4} A^+A^- + e_0 = e^{-\omega/4} A^-A^+ - e_0 = 2\left(\hat{T}^2 - \cosh \frac{\omega}{4} \right) \end{aligned} \tag{66}$$

where

$$e_0 = 2\sinh\frac{\omega}{4}. \tag{67}$$

It can easily be shown that

$$\mathrm{e}^{\omega/8} A^{\pm}\hat{T} - \mathrm{e}^{-\omega/8}\hat{T}A^{\pm} = [\hat{A}^{\pm}, \hat{T}]_{q^{1/2}} = 0. \tag{68}$$

It follows from (65) and (66) that

$$[A^{+}, h]_q = (q^2 - q^{-2})A^{+} \tag{69}$$

$$[A^{-}, h]_{q^{-1}} = -(q^2 - q^{-2})A^{-}. \tag{70}$$

This justifies the interpretation of $A^{\pm}$ as the creation and annihilation operators. Putting

$$\psi_{n+1}(x) = \left(\exp\frac{n+1}{2}\omega 4\sinh\frac{n+1}{2}\omega\right)^{-1/2} A^{+}\psi_n(x) \tag{71}$$

we obtain from (69) and (41) the recurrence relations connecting the neighbouring levels,

$$\mathrm{e}_{n+1} = \mathrm{e}^{\omega/2}\,\mathrm{e}_n + 2\,\mathrm{e}^{\omega/4}\sinh\frac{\omega}{2} \tag{72}$$

which gives the spectral formula

$$\mathrm{e}_n = 2\left(\exp\frac{2n+1}{4}\omega - \cosh\frac{\omega}{4}\right). \tag{73}$$

Notice that this spectral formula also corresponds to the Hamiltonian

$$\hat{h}_1 = 2mc^2\left(\exp\left(\frac{H_{\mathrm{nr}}}{2mc^2}\right) - \cosh\frac{\hbar\omega}{4mc^2}\right) \tag{74}$$

where H_{nr} is the non-relativistic Hamiltonian and the dimensional variables are restored.

The eigenfunctions of the finite-difference equation (41) with the Hamiltonian given by (66) are defined in the form

$$\psi_n(x) = N_n\,\mathrm{e}^{-\omega x^2/2}\,h_n(x) \tag{75}$$

$$N_n = \left[\left(\frac{\pi}{\omega}\right)^{1/2} 2^n\,\mathrm{e}^{-\omega/16}[n]_q\,!\right]^{-1/2} \tag{76}$$

where by definition

$$[n]_q\,! = \left[\left(\frac{4}{\omega}\right)^n \prod_{k=1}^{n}\exp\frac{k\omega}{4}\sinh\frac{k\omega}{4}\right]. \tag{77}$$

The wavefunctions obey the normalization conditions

$$\int_{-\infty}^{\infty}\psi_n(x)\psi_m(x)\cos\frac{\omega x}{2}\,\mathrm{d}x = \delta_{nm}. \tag{78}$$

In (75) the $h_n(x)$ are the 'relativistic Hermite polynomials' which we must find. They satisfy the equation

$$\left(\left(\mathrm{e}^{\omega x^2/2}\,\hat{T}\,\mathrm{e}^{-\omega x^2/2}\right)^2 - 2\cosh\frac{\omega}{4} - \mathrm{e}_n\right)h_n(x) = 0 \tag{79}$$

where $\hat{T}$ is given by (63). When constructing the theory of the h_n functions we shall follow the scheme accepted for the non-relativistic oscillatory solutions taking into account the finite-difference character of the formalism.

The $h_n(x)$ are defined by the 'Rodriguez formula':

$$h_n(x) = -\frac{1}{\sqrt{\omega}} e^{\omega x^2} \hat{\mathscr{D}} e^{-\omega x^2} h_{n-1} \tag{80}$$

$$h_n(x) = \left(-\frac{1}{\sqrt{\omega}}\right)^n e^{\omega x^2} \hat{\mathscr{D}}^n e^{-\omega x^2} \tag{81}$$

$$h_0(x) = 1. \tag{82}$$

The $h_n(x)$ are polynomials of nth degree of the variable $\sin \omega x/2$. They satisfy the following recurrence relations:

$$h_n(-x) = (-1)^n h_n(x) \tag{83}$$

$$\sqrt{\omega}\, \hat{\mathscr{D}} h_n(x) = 8\, e^{(n/4)\omega} \sinh \frac{h\omega}{4} h_{n-1}(x) \tag{84}$$

$$h_{n+1}(x) - \frac{4}{\sqrt{\omega}} e^{[(n+1)/4]\omega} \sin \frac{\omega x}{2} h_m(x) + \frac{8}{\omega} \sinh \frac{h\omega}{4} e^{[(n+1)/4]\omega} H_{n-1}(x) = 0 \tag{85}$$

$$e^{(n/4)\omega} h_n(x) - \cosh \frac{i}{2} \frac{d}{dx} h_n(x) + i \tan \frac{\omega x}{2} \sinh \frac{i}{2} \frac{d}{dx} h_n(x) = 0. \tag{86}$$

We can also derive relations analogous to

$$\begin{aligned} m!\, H_{2m}(x) &= (-1)^m (2m)!\, {}_1F_1(-m; \tfrac{1}{2}; x^2) \\ m!\, H_{2m+1}(x) &= (-1)^m (2m+1)!\, 2x\, {}_1F_1(-m; \tfrac{3}{2}; x^2) \end{aligned} \tag{87}$$

expressing the Hermite polynomials in terms of confluent hypergoemetric functions. We have

$$h_{2m}(x) = \sum_{l=0}^{m} \alpha_l^{2m} \cos l\omega x \tag{88}$$

where

$$\alpha_l^{2m} = \alpha_0^{2m} 2(-1)^l e^{\omega l^2/4} \frac{\sinh \frac{m\omega}{4} \sinh \frac{m-1}{4}\omega \ldots \sinh \frac{m-l+1}{4}\omega}{\sinh \frac{m+1}{4}\omega \sinh \frac{m+2}{4}\omega \ldots \sinh \frac{m+l}{4}\omega}$$

$$\alpha_0^{2m} = \frac{4^m e^{[m(m+1)/4]\omega}}{\omega^m} \frac{\sinh \frac{m+1}{4}\omega \sinh \frac{m+2}{4}\omega \ldots \sinh \frac{2m-1}{4}\omega \sinh \frac{2m}{4}\omega}{\sinh \frac{\omega}{4} \sinh \frac{2\omega}{4} \ldots \sinh \frac{m\omega}{4}}$$

and

$$h_{2m+1} = \sum_{l=0}^{m} \beta_l^{2m+1} \sin(l+\tfrac{1}{2})\omega x \tag{89}$$

where

$$\beta_l^{2m+1}=\beta_0^{2m+1}(-1)^l\,e^{[l(l+1)/4]\omega}\,\frac{\sinh\frac{m\omega}{4}\sinh\frac{m-1}{4}\omega\ldots\sinh\frac{m-l+1}{4}\omega}{\sinh\frac{m+2}{4}\omega\sinh\frac{m+3}{4}\omega\ldots\sinh\frac{m+l+1}{4}\omega}$$

$$\beta_0^{2m+1}=\frac{\sqrt{\omega}}{2}\,e^{-[(m+1)/4]\omega}\cosh\frac{m+1}{4}\omega.$$

The integral representation for the relativistic Hermite polynomials can be derived by applying the recurrence relation (81) to the identity

$$e^{-\omega x^2}=\frac{1}{\sqrt{\pi\omega}}\int_{-\infty}^{\infty}e^{-t^2/\omega}\,e^{2ixt}\,dt. \tag{90}$$

We have

$$e^{-\omega x^2}h_n(x)=\frac{1}{\sqrt{\pi\omega}}\left(\frac{e^{-i\pi/2}\,2}{\sqrt{\omega}}\right)^n\int_{-\infty}^{\infty}T_n(t)\,e^{2ixt}\,e^{-t^2/\omega}\,dt \tag{91}$$

where $T_n(t)$ are the polynomials which satisfy the recurrence relations

$$e^{-t^2/\omega}\,T_{n+1}(t)+2\,e^{(n+1)/4}\sinh\frac{\omega}{4}\frac{d}{dt}(e^{-t^2/\omega}\,T_n(t))-2\sinh\frac{h\omega}{4}$$
$$\times e^{[(n+1)/4]\omega}\,e^{-t^2/\omega}\,T_{n-1}(t)=0 \tag{92}$$

$$T_{n+1}(t)=\frac{e^{[(2n+3)/16]\omega}(\cosh t-\cosh\frac{1}{4}h\omega)}{\sinh\frac{1}{8}h\omega\cosh\frac{1}{2}t}\sinh\frac{\omega}{4}\frac{d}{dt}T_n(t) \tag{93a}$$

$$T_{n+1}(t)=\frac{e^{[(2n+3)/16]\omega}(\cosh t-\cosh\frac{1}{4}h\omega)}{\cosh\frac{1}{8}h\omega\sinh\frac{1}{2}t}\cosh\frac{\omega}{4}\frac{d}{dt}T_n(t). \tag{93b}$$

The solution of these recurrence relations is

$$T_{2m}(t)=2^m\,e^{[m(m+1)/4]\omega}\prod_{l=0}^{m-1}\left(\cosh t-\cosh\frac{2l+1}{4}\omega\right) \tag{94}$$

$$T_{2m+1}(t)=2^{m+1}\,e^{[4(m+1)^2-1/16]\omega}\sinh\frac{t}{2}\prod_{l=0}^{m-1}\left(\cosh t-\cosh\frac{l+1}{2}\omega\right). \tag{95}$$

5. The generalized Heisenberg–Weyl group and relativistic coherent states

As the consequence of the finite-difference calculus rule (46) the canonical commutation relation of coordinate x and momentum $\hat{k}$ (39) is changed as compared with non-relativistic quantum mechanics. We have

$$[\hat{x},\hat{k}]=i\hbar\hat{A} \tag{96}$$

where

$$\hat{A}=\cosh\frac{i\hbar}{2mc}\frac{d}{dx}. \tag{97}$$

We stress that operators $\hat{k}$ and $\hat{A}$ belong either to the universal enveloping algebra of the usual translations or to the non-relativistic Heisenberg–Weyl group. Furthermore

$$[\hat{k}, \hat{A}] = 0 \qquad [\hat{x}, \hat{A}] = -\frac{i\hbar}{(2mc^2)}\hat{k}. \tag{98}$$

In the non-relativistic limit

$$\hat{A} \to \hat{I} \qquad \text{(unity operator)} \tag{99}$$

and relations (96) and (98) reduce to

$$[x, \hat{k}] = i\hbar\, \hat{I} \qquad [\hat{k}, \hat{I}] = [\hat{x}, \hat{I}] = 0 \tag{100}$$

which is the well known Hesienberg–Weyl group. It is then natural to call the set of commutation relations (96) and (98) the generalized or relativistic Heisenberg–Weyl group. This group can be extended, including the unity operator,

$$[x, \hat{I}] = [\hat{k}, \hat{I}] = [\hat{A}, \hat{I}] = 0. \tag{101}$$

The extended group can also be considered as the generalization of the Heisenberg–Weyl group, to which it tends in the non-relativistic limit. On the right-hand side of (96) we have an operator, instead of a number as in the non-relativistic case. The relativistic uncertainties relation has the form

$$\overline{(\Delta\hat{k}^2)}\,\overline{(\Delta x^2)} \geqslant \tfrac{1}{4}(\bar{\hat{A}})^2. \tag{102}$$

Notice in this connection (Celeghini *et al* 1990) where problems connected with the q-deformation of the canonical commutation relation and squeezing of light are considered. We look for the coherent states which minimize this relation, i.e. the states for which the equality in (102) holds.

Let us consider the equation

$$(\hat{x} + i\mu\hat{k} - \Delta)\psi(x) = 0 \tag{103}$$

where

$$\Delta = \bar{x} + i\mu\bar{k} \tag{104}$$

and μ is real parameter, $\bar{x}$ and $\bar{k}$ are the average values of x and $\hat{k}$ correspondingly.

The solution of (103) can be found using the relativistic Fourier expansion

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \langle x|\lambda\rangle \tilde{\psi}(\lambda)\, d\lambda \tag{105}$$

where λ is the rapidity. In terms of rapidities the finite-difference equation (103) takes the form of a first-order differential equation:

$$\left(i\frac{d}{d\lambda} + 2i\mu\sinh\frac{\lambda}{2} - \Delta\right)\tilde{\psi}(\lambda)\, d\lambda = 0. \tag{106}$$

The solution to this equation is

$$\tilde{\psi}(\lambda) = c\, e^{-4\mu\cosh\lambda/2 - i\Delta\lambda}. \tag{107}$$

This is the kernel of the integral representation for the Macdonald function

$$K_\nu(z) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-z\cosh t + \nu t}\, dt \tag{108}$$

and we obtain

$$\psi(x)=\frac{4c}{\sqrt{2\pi}}K_{2i(x-\Delta)}(4\mu). \tag{109}$$

Now taking into account the recurrence relation for $K_\nu n(z)$,

$$K_{\nu-1}(z)-K_{\nu+1}(z)=-\frac{2\nu}{z}K_\nu(z) \tag{110}$$

it is easy to verify that (109) is indeed the solution of the finite-difference equation (103). The normalization integral for ψ is calculated with the help of the integral representation (108) and the normalized ψ has the form

$$\psi(x)=\left(\frac{\pi K_{3\mu\bar{k}}(8\mu)}{2}\right)^{1/2}K_{2i(x-\Delta)}(4\mu). \tag{111}$$

Let us compute now the quadratic average values of momentum and coordinate operators and average value of $\hat{A}$. We have

$$\langle\hat{k}^2\rangle=\int_{-\infty}^{\infty}\psi^*(x)\left(-2\sinh\frac{i}{2}\frac{d}{dx}\right)^2\psi(x)\,dx. \tag{112}$$

The following finite-difference analogue of partial integration can be proved:

$$\int_{-\infty}^{\infty}f(x)\sinh\frac{i}{2}\frac{d}{dx}\varphi(x)\,dx=-\int_{-\infty}^{\infty}\left(\sinh\frac{i}{2}\frac{d}{dx}f(x)\right)\varphi(x)\,dx \tag{113}$$

provided that $f(x)$ and $\varphi(x)$ vanish at infinity sufficiently rapidly for all integrals to exist. This gives

$$\langle\hat{k}^2\rangle=-4\int\left(\sinh\frac{i}{2}\frac{d}{dx}\psi^*(x)\right)\left(\sinh\frac{i}{2}\frac{d}{dx}\psi(x)\right)dx. \tag{114}$$

Now, using the explicit expression for ψ (111) and (110) we have

$$\langle\hat{k}^2\rangle=\frac{1}{\mu^2}\int(x^2+\mu^2\bar{k}^2)\psi^*(x)\psi(x)\,dx=\frac{1}{\mu^2}\langle x^2\rangle+\bar{K}^2. \tag{115}$$

The expression for the average quadratic value of a coordinante is

$$\langle x^2\rangle=\int_{-\infty}^{\infty}\bar{\psi}(x)x^2\psi(x)\,dx \tag{116}$$

$$\begin{aligned}
&=\left(\frac{4c}{\sqrt{2\pi}}\right)^2\int_{-\infty}^{\infty}K_{2i(x-\Delta)}(4\mu)x\left(2\,i\mu\sinh i\frac{d}{dx}+i\mu\bar{k}\right)K_{2i(x-\Delta)}(4\mu)\,dx\\
&=i\mu\bar{k}\left(\frac{4c}{\sqrt{2\pi}}\right)^2\int_{-\infty}^{\infty}K_{2i(x-\Delta)}(4\mu)\times xK_{2i(x-\Delta)}(4\mu)\,dx\\
&\quad-2i\mu\left(\frac{4c}{\sqrt{2\pi}}\right)^2\int_{-\infty}^{\infty}\left(\sinh\frac{i}{2}\frac{d}{dx}[xK_{2i(x-\Delta)}(4\mu)]\right)\times K_{2i(x-\Delta)}(4\mu)\,dx
\end{aligned} \tag{117}$$

where (113) was used again. The first integral in this expression is zero in consequence of oddness of the integrand. In the second integral we again use the partial integration rule (113). We have

$$\langle x^2\rangle = \mu\left(\frac{4c}{2\pi}\right)^2 \int_{-\infty}^{\infty} \left(\cosh\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}x} K_{2\mathrm{i}(x-\bar{\Delta})}(4\mu)\right) \times K_{2\mathrm{i}(x\Delta)}((4\mu)\,\mathrm{d}x$$

$$-\left(\frac{4c}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} x(x-\mathrm{i}\bar{\Delta}) K_{2\mathrm{i}(x-\Delta)}(4\mu) K_{2\mathrm{i}(x-\Delta)}(4\mu)\,\mathrm{d}x \tag{118}$$

which after omitting the vanishing integral finally gives

$$\langle x^2\rangle = \mu\left\langle \cosh\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle - \langle x^2\rangle. \tag{119}$$

It follows that

$$\langle x^2\rangle = \frac{\mu}{2}\left\langle \cosh\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle = \frac{\mu}{2}\langle\hat{\mathscr{A}}\rangle. \tag{120}$$

The average value of $\hat{A}$ is calculated using the recurrence relation

$$K_{\nu-1}(z) + K_{\nu+1}(z) = -2\frac{\mathrm{d}}{\mathrm{d}z}K_\nu(z) \tag{121}$$

which gives

$$\langle\hat{A}\rangle = \left(\frac{4c}{\sqrt{2\pi}}\right)^2 \int K_{2\mathrm{i}(x-\Delta)}(4\mu)\cosh\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}x} K_{2\mathrm{i}(x-\Delta)}(4\mu)\,\mathrm{d}x$$

$$= -\left(\frac{4c}{\sqrt{2\pi}}\right)^2 \frac{\mathrm{d}}{\mathrm{d}z}\int K_{2\mathrm{i}(x-\Delta)}(4\mu) K_{2\mathrm{i}(x-\Delta)}(z)\,\mathrm{d}x\Big|_{z=4\mu}$$

$$= -\frac{\pi}{2}\left(\frac{4c}{\sqrt{2\pi}}\right)^2 \frac{\mathrm{d}}{\mathrm{d}z} K_{4\mu\bar{\kappa}}(4\mu+z)\Big|_{z=4\mu}. \tag{122}$$

Taking (120) into account we can rewrite (115) in the form

$$\langle\hat{k}^2\rangle = \frac{1}{2\mu}\langle\hat{\mathscr{A}}\rangle + \bar{k}^2. \tag{123}$$

Then

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle = \frac{\mu}{2}\langle\hat{\mathscr{A}}\rangle \tag{124}$$

$$\langle(\Delta k)^2\rangle = \langle k^2\rangle - \bar{k}^2 = \frac{1}{2\mu}\langle\hat{\mathscr{A}}\rangle \tag{125}$$

and finally we arrive at the equality

$$\langle(\Delta x)^2\rangle\langle(\Delta\hat{k})^2\rangle = \tfrac{1}{4}\langle\hat{\mathscr{A}}^2\rangle \tag{126}$$

which shows that our coherent states (109) indeed minimize the uncertainty relation (102).

6. The dynamical symmetry of the relativistic oscillator

It is easy to show that the operators

$$L^+ = \frac{1}{2\delta_-\delta_+^{1/2}}(\mathscr{A}^+\mathscr{A}^-)^{1/2}\mathscr{A}^+ \qquad L^- = \frac{1}{2\delta_-\delta_+^{1/2}}\mathscr{A}^-(\mathscr{A}^+\mathscr{A}^-)^{1/2} \qquad L^3 = \frac{1}{\delta_+\delta_-}\hat{h} \tag{127}$$

where

$$\delta_\pm = (q^{-1} \pm q) \tag{128}$$

obey the deformed (quantum) Lie algebra $SU_q(1,1)$ relations (Kagramanov *et al* 1989, Chaichian and Kulish 1990, Chaichian *et al* 1990)

$$[L^+, L^-]_{q^2} = L^3 \qquad [L^\pm, L^3]_{q^{\pm 1}} = \mp L^\pm. \tag{129}$$

In the non-relativistic limit the expressions (127) and (129) turn into the well known formulae for the generators of the group SU(1, 1) of dynamical symmetry of non-relativistic oscillator (see, for example, Malkin and Manko 1979). for the deformed Lie q-algebra (129) the deformed symmetry and Jacobi relations are fulfilled

$$[A, B]_q = -[B, \mathscr{A}]_{q^{-1}} \tag{130}$$

$$[[A, B]_q, C] + [[B, C]_q, A] + [[C, A]_q, B] = 0. \tag{131}$$

Let us introduce the operators

$$\hat{a}^+ = (2\delta_-)^{-1/2}\mathscr{A}^+ q^{N/2} \tag{132}$$

$$\hat{a}^- = (2\delta_-)^{-1/2} q^{N/2}\mathscr{A}^- \tag{133}$$

where

$$N = -\frac{\ln(1 + \frac{1}{2}\mathscr{A}^+\mathscr{A}^-)}{2\ln q}. \tag{134}$$

We have

$$\hat{a}^-\hat{a}^+ - q^{-1}\hat{a}^+\hat{a}^- = q^N \tag{135a}$$

$$\hat{a}^-\hat{a}^+ - q\hat{a}^+\hat{a}^- = q^{-N}. \tag{135b}$$

The operator N is connected with the oscilltor Hamiltonian (66) and $\hat{T}$ in (63) by the relations

$$\hat{h} = 2\left(q^{-(2N+1)} - \frac{q + q^{-1}}{2}\right) \tag{136}$$

$$T = q^{-N-1/2}. \tag{137}$$

It follows from (68), (133) and (137) that

$$[N, \hat{a}^\pm] = \pm\hat{a}^\pm \tag{138}$$

and

$$q^{rN}\hat{a}^+ q^{-rN} = q^r\hat{a}^+ \tag{139a}$$

$$q^{rN}\hat{a}^- q^{-rN} = q^{-r}\hat{a}^-. \tag{139b}$$

The relations (135) can also be written in the form

$$\hat{a}^+\hat{a}^- = [N] \tag{140a}$$

$$\hat{a}^-\hat{a}^+ = [N+1] \tag{140b}$$

where

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sinh \omega/4x}{\sinh \omega/4}. \tag{141}$$

Now defining new q-generators $J^{\pm}$, J^3 as

$$J^+ = \frac{1}{\sqrt{2}}(\hat{a}^+\hat{a}^-)^{1/2}\hat{a}^+ \qquad J^- = \frac{1}{\sqrt{2}}\hat{a}^-(\hat{a}^+\hat{a}^-)^{1/2} \qquad J^3 = 2N+1 \tag{142}$$

we come to C-algebra (see Abe 1980) in the form considered by Jimbo (1987), Macfarlane (1989), Biedenharn (1989), Woronowicz (1988), Hayashi (1990) and Floreani *et al* (1990):

$$[J^3, J^{\pm}] = \pm J^{\pm} \qquad [J^+, J^-] = -[J^3]. \tag{143}$$

We preferred here the construction for L- and J-operators which is a modification of the one described for example by Barut and Fronsdal (1965), Barut and Girardello (1971) and Malkin and Manko (1979) instead of the Jordan–Schwinger mapping used by Macfarlane (1989) and Biedenharn (1989).

One of the approaches to quantum groups defines them by the matrices of the lowest-order representations (Faddeev 1984, Manin 1987, Vokos *et al* 1989). Thus the simplest example of such a method is the definition of the *quantum two-dimensional* linear group $SL_q(2, C)$ by giving quantization relations for the elements of the 2×2 SL(2, C) matrices. We say that a 2×2 matrix

$$\mathscr{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{144}$$

belongs to the quantum group $GL_q(2, C)$ if its matrix elements, instead of being *complex numbers, are non-commuting quantities satisfying the commutation relations*

$$\begin{aligned} &ab = qba \\ &ac = qca \qquad bc = cb \\ &bd = qdb \qquad ad - da = (q - q^{-1})bc \\ &cd = qdc. \end{aligned} \tag{145}$$

We shall show here that the quantities a, b, c, d are the finite-difference operators acting in Hilbert space and they can be expressed in terms of $A^{\pm}$ and $\hat{T}$ introduced by the relations (56) and (63), Namely, taking into account the relations (65), (66) and (68) we see that the elements of the matrix

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}^+/\sqrt{2} & \hat{T} \\ -\hat{T} & -\mathscr{A}^-/\sqrt{2} \end{pmatrix} \tag{146}$$

obey all the conditions listed in (145). As stressed by Vokos *et al* (1989), to formulate the composition law for such matrices we must consider a manifold of matrices of the same type, i.e. with elements satisfying the commutation relations

$$\begin{aligned} &a'b' = qb'a' \\ &a'c' = qc'a' \qquad b'c' = c'b' \\ &b'd' = qd'b' \qquad a'd' - d'a' = (q - q^{-1})b'c' \\ &c'd' = qd'c'. \end{aligned} \tag{147}$$

We must also suppose that a', b', c', d' commute with a, b, c, d. In the language of our relativistic oscillator this means that we must consider an infinite number of independent oscillators. It is easy to show that the matrix

$$\mathscr{A}'' = \mathscr{A}\mathscr{A}' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & ca' + dd' \end{pmatrix} \tag{148}$$

is the same type as $\mathscr{A}$ and $\mathscr{A}'$, i.e.

$$\begin{aligned} &a''b'' = ab''a'' \\ &a''c'' = qc''a'' \qquad b''c'' = c''b'' \\ &b''d'' = qd''b'' \qquad a''d'' - d''a'' = (q - q^{-1})b''c'' \\ &c''d'' = qd''c''. \end{aligned} \tag{149}$$

The quantum determinant of the matrix $\mathscr{A}$ is defined as

$$D_q = \det_q \mathscr{A} = ad - qbc = da - \frac{1}{q}bc. \tag{150}$$

It reduces to the usual determinant for $q = 1$. Using the conditions (145) it can easily be seen that D_q commutes with the elements a, b, c, d. Again using (66) we find that the quantum determinant of the matrix (146) is unity:

$$\det_q \mathscr{A} = 1. \tag{151}$$

The inverse matrix (both left and right) is

$$A^{-1} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} d & -(1/q)b \\ -qc & a \end{pmatrix} = \begin{pmatrix} -A^-/\sqrt{2} & -1/qT \\ qT & A^+/\sqrt{2} \end{pmatrix}. \tag{152}$$

This matrix corresponds to the quantization parameter q^{-1} (or $-\omega$), for example

$$\tilde{a}\tilde{b} = d\left(-\frac{1}{q}b\right) = -\frac{1}{q}\left(\frac{1}{q}bd\right) = \frac{1}{q}\tilde{b}\tilde{a}. \tag{153}$$

The conditions (145) can be considered as a quantum generalization of symplectivity conditions for the matrix A. We can say that the 'transformations' generated by the matrix $\mathscr{A}$ conserve the symplectic form

$$\xi = \bar{x}\varepsilon_q y = 1/\sqrt{q} \tag{154}$$

where

$$\bar{x} = \left(\frac{A^+}{\sqrt{2}}, -T\right) \qquad y = \begin{pmatrix} T \\ -A^-/\sqrt{2} \end{pmatrix} \tag{155}$$

and the quantum metric tensor is

$$\varepsilon_q = \begin{pmatrix} 0 & 1/\sqrt{q} \\ -\sqrt{q} & 0 \end{pmatrix} \qquad \varepsilon_q^2 = -1. \tag{156}$$

Indeed, we can easily verify that

$$\varepsilon_q A^{\mathrm{T}} \varepsilon_q^{-1} = \mathcal{A}^{-1} \tag{157}$$

where $\mathcal{A}^{\mathrm{T}}$ is the transpose of the matrix $\mathcal{A}$.

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